Chapter 11.2 part 3

Corollary 11.8 and its proof

On Corollary 11.8 and its proof.

Let $\sigma: F \to E$ be a field isomorphism.

<u>Prop</u> "σ <u>extends</u> to an isomorphism of the polynomial rings"

The map $\sigma: F[x] \to E[x]$

$$a_0 + \dots + a_m x^m \longmapsto \sigma(a_0) + \dots + \sigma(a_m) x^m$$

$$a \longmapsto \sigma(a) \quad \text{for } a \in F$$

$$x \longmapsto x$$

is a ring isomorphism

$F \subset F[x]$

$$\begin{tikzcd} F[x] \arrow[r, "\sigma"] & E[x] \\ F \arrow[u, dash] \arrow[r, "\sigma"'] & E \arrow[u, dash] \end{tikzcd}$$

Notation: for a polynomial $f = a_0 + \dots + a_m x^m \in F[x]$ we write

$$\sigma f = \sigma(a_0) + \sigma(a_1) x + \dots + \sigma(a_m) x^m \in E[x]$$

for the image of f under σ (σf instead of $\sigma(f)$)

Th 11.7 (1) Let $K \supset F$ and, for $u \in K$ let $p \in F[x]$ be the minimal polynomial of u.

Then $F[x]/(p) \simeq F(u)$

Remark. Let $v \in K$ such that $p \in F[x]$ is the min poly also for v. Then $F(v) \simeq F(u)$

From the proof:

$$\varphi : F[x] \longrightarrow F(u) \qquad \ker \varphi = (p) \qquad \varphi \text{ is surjective}$$
$$f \longmapsto f(u)$$

the isomorphism:

$$\bar{\varphi} : F[x]/(p) \xrightarrow{\sim} F(u)$$
$$[f] \longmapsto f(u)$$

Similarly, consider $\sigma p \in E[x]$, let σp be the minimal polynomial for $v \in L$ for a field extension $L \supset E$.

$$\tau : E[x] \longrightarrow E(v)$$
$$f \longmapsto f(v)$$

$$\bar{\tau} : E[x]/(\sigma p) \xrightarrow{\sim} E(v)$$
$$[f] \longmapsto f(v)$$

Cor 11.8 $\sigma : F \to E$ is a field isomorphism

u is algebraic over F with its min poly $p \in F[x]$

v —— " —— E —— " —— $\sigma p \in E[x]$

Then there exists $\tilde{\sigma} : F(u) \to E(v)$ - field isomorphism such that $\tilde{\sigma}(c) = \sigma(c)$ for every $c \in F$

$\tilde{\sigma}(u) = v$

" $\tilde{\sigma}$ extends σ "

$$\begin{tikzcd} F(u) \arrow[r, "\tilde{\sigma}"', "u \mapsto v"] \arrow[d, dash] & E(v) \arrow[d, dash] \\ F \arrow[r, "\sigma"'] & E \end{tikzcd}$$

Pf

Notation $\pi : E[x] \to E[x]/(\sigma p)$

$f \mapsto [f]$

$\pi : R \to R/I$

π is surjective

We have constructed so far:

$$F[x] \xrightarrow[\simeq]{\sigma} E[x] \underset{\text{surjective}}{\overset{\pi}{\twoheadrightarrow}} E[x]/(\sigma p) \xrightarrow[\simeq]{\bar{\tau}} E(v)$$

The composition $\bar{\tau} \circ \bar{\jmath} \circ \sigma$ is surjective (because all three maps are surjective)

The kernel consists of $h \in F[x]$ such that $(\sigma h)(v) = 0_E$

iff $\sigma p \mid \sigma h$

$\sigma h = k \, \sigma p \quad k \in E[x]$

$h = \sigma^{-1} k \cdot p \quad \sigma^{-1} k \in F[x]$

$h \in F[x]$ is in the kernel of the composition map $\bar{\tau} \circ \bar{\jmath} \circ \sigma$ iff h is a multiple of p ($\sigma^{-1} k$ is an arbitrary polynomial in $F[x]$)

The kernel $\operatorname{Ker}(\bar{\tau} \circ \bar{\jmath} \circ \sigma) = (p)$

$\bar{\tau} \circ \bar{\jmath} \circ \sigma : F[x] \longrightarrow E(v)$ is surjective

The first Isom-m Thm implies the isomorphism

$$\Theta : F[x]/(p) \xrightarrow{\simeq} E(v)$$

$$[f] \longmapsto (\sigma f)(v)$$

$$[x] \longmapsto v$$

notation

We know from Th 11.7(1)

$$F[x]/(p) \simeq F(u)$$

Thus $E(v) \simeq F(u)$.

$$
\begin{array}{ccccc}
F(u) & \xleftarrow[\simeq]{\bar{\varphi}} & F[x]/(p) & \xrightarrow[\simeq]{\Theta} & E(v) \\
f(u) & \longleftarrow\!\shortmid & [f] & \longmapsto & (\sigma f)(v) \\
c & \longleftarrow\!\shortmid & [c] & \longmapsto & \sigma(c) \\
u & \longleftarrow\!\shortmid & [x] & \longmapsto & v
\end{array}
\qquad c \in F
$$

$$\bar{\sigma} = \Theta \circ \bar{\varphi}^{-1} : F(u) \longrightarrow E(v)$$

$$\bar{\sigma}(c) = \sigma(c)$$

$$\bar{\sigma}(u) = v$$